

Network models and systemic risk assessment

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Abstract

During the last years a number of network models of interbank markets were developed and applied to the analysis of insolvency contagion and systemic risk. In this chapter we survey the concepts used in these models and discuss their main findings as well as their applications in systemic risk analysis. Network models are designed to address potential domino effects resulting from the failure of a financial institutions. Specifically they attempt to answer the question whether the failure of an institution will result in the subsequent failure of others. Since in a banking crisis authorities usually intervene to stabilize the banking system failures and contagious failures by domino effects are very rarely observed in practice. Empirical analysis is thus difficult and as a consequence most studies of insolvency contagion built on simulation models. In this chapter we describe in some detail how such simulations are designed and discuss the main insights that have so far been obtained by applications to the complex network of real world exposure data of banking systems.

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JEL-Classification Numbers: G21, C15, C81, E44

1 Introduction

Will the failure of a financial institution be a threat to the stability of the banking system? This is the key question for authorities in the management of a financial crisis. At the height of a crisis the general level of uncertainty and the panic among market participants usually lead to stabilization policies and interventions of the public sector. Therefore the unfolding of default cascades and the realization of domino effects of insolvency are rarely observed and there is no reasonable database that would allow a systematic and reliable empirical answer to the question of how big contagion risks actually are. Against this background during the past decade a number of simulation models were developed that try to find a way to analyze contagion risk and domino effects in a more systematic way.

In this chapter we give an overview about the concepts and tools that have been used in this literature. We should say that the contagion channel analyzed in these papers is narrowly concerned with domino effects of insolvency that result from the complex network of debt contracts between financial institutions. In the literature many other spillover channels are studied and referred to as contagion. Upper (2011) contains a systematic overview about different contagion channels. In this chapter we refer to contagion as domino effects of the insolvency of banks.

We start with a description of network models of banking systems in section (2). In particular we explain how potential insolvencies of particular banks in the system are consistently resolved by the calculation of clearing payment vectors. In section (3) we describe in some detail how the network of interbank debt exposures can be estimated from incomplete data. This is an important issue in applications because only in very rare cases data will allow to reconstruct the bilateral debt exposures exactly. In most of the cases the data on the network of liabilities will have to be - at least partially - estimated. Since most papers in the literature are only very sketchy in describing how this is actually done, we describe the estimation procedure in some detail. An important element of a simulation model of contagion will be the simulation of loss scenarios. We describe loss scenario generation in section (4). The techniques applied in the generation of loss scenarios build mainly on standard techniques used in risk management. As a consequence we are relatively brief in this section and refer for technical details mostly to the literature. Section (5) describes the clearing procedure and discusses two approaches to the calculation of clearing vectors. While the first approach due to Eisenberg and Noe (2001) is known from the literature as the fictitious default algorithm, the second procedure is not so

widely known but has the advantage that it can be applied to more general clearing situations where the fictitious default algorithm fails. In section (6) we give a summary of the results most simulation studies yielded in applications to real world bank exposure data. Since these findings have been excellently surveyed in Upper (2011) we can be relatively brief. Section (7) discusses extensions and concludes.

2 A Network Model of Interbank Exposures and Contagion Risk

Consider an economy populated by n banks constituting a *financial network*. The asset side of the balance sheet of bank i consists of non interbank related assets a_i^{NIB} and of interbank assets a_i^{IB} . On the liabilities side we find interbank liabilities d_i^{IB} as well as liabilities to creditors outside the network d_i^{NIB} and of course as a residual equity e_i . The value of the non interbank assets a_i^{NIB} is interpreted as an exogenous random variable. The values of the remaining parts of the balance sheet are determined endogenously within the network conditional on a particular draw of $a^{NIB} = (a_1^{NIB}, \dots, a_n^{NIB})'$.

Not all the liabilities will be of the same seniority and some of the interbank assets may be more senior than others. The network model is able to take this correctly into account as is shown in Elsinger (2009). To keep the description of the model as simple as possible we assume in this context that there is only one seniority class.¹

The structure of the interbank liabilities is represented by an $n \times n$ matrix L where l_{ij} represents the nominal obligation of bank i to bank j . These liabilities are nonnegative and the diagonal elements of L are zero as banks are not allowed to hold liabilities against themselves. Evidently,

$$\sum_{j=1}^n l_{ij} = \bar{d}_i^{IB} \quad \text{and} \quad \sum_{i=1}^n l_{ij} = \bar{a}_j^{IB} \quad (1)$$

where \bar{d}_i^{IB} and \bar{a}_j^{IB} denote the nominal values of the interbank claims and liabilities in contrast to the endogenously determined market values d_i^{IB} and a_j^{IB} .

A bank is defined to be in default whenever exogenous income plus the amounts received

¹We could adopt the definition of a_i^{NIB} and include all interbank claims and liabilities except the most junior into a_i^{NIB} . But this would lead to inconsistencies in case of default.

from other nodes are insufficient to cover the bank's nominal liabilities.² Bank defaults do not change the prices outside of the network, i.e. a^{NIB} is independent of defaults and exogenous.

In case of default the clearing procedure has to respect three criteria:

1. limited liability: which requires that the total payments made by a node must never exceed the sum of exogenous income and payments received from other nodes,
2. priority of debt claims: which requires that stockholders receive nothing unless the bank is able to pay off all of its outstanding debt completely, and
3. proportionality: which requires that in case of default all claimant nodes are paid off in proportion to the size of their claims on firm assets.

To operationalize proportionality let \bar{p}_i be the total nominal obligations of node i , i.e.

$$\bar{d}_i = \bar{d}_i^{IB} + \bar{d}_i^{NIB} = \sum_{j=1}^n l_{ij} + \bar{d}_i^{NIB}$$

and define the proportionality matrix Π by

$$\Pi_{ij} = \begin{cases} \frac{l_{ij}}{\bar{d}_i} & \text{if } \bar{d}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Evidently, it has to hold that $\Pi \cdot \vec{1} \leq \vec{1}$.

To simplify notation we define for any two (column) vectors $x, y \in \mathbb{R}^n$ the lattice operations

$$\begin{aligned} x \wedge y &:= (\min(x_1, y_1), \dots, \min(x_n, y_n))' \\ x \vee y &:= (\max(x_1, y_1), \dots, \max(x_n, y_n))'. \end{aligned}$$

Let $d = (d_1, \dots, d_n)' \in \mathbb{R}_+^n$ be an arbitrary vector of payments made by banks to their interbank and non interbank creditors. The equity values E of the banks may be defined as

$$E(d) = [a^{NIB} + \Pi' d - d] \vee \vec{0}. \quad (2)$$

²A bank is in default if liabilities exceed assets. Using a violation of capital requirements as default threshold does not change the main results.

However, for an arbitrary d it is possible that the equity value of bank i is positive ($E_i(d) > 0$) but the actual payments made do not cover the liabilities ($d_i < \bar{d}_i$). In this case absolute priority would not hold. For a given d the amount available for bank i to pay off its debt equals $a_i^{NIB} + \sum_{j=1}^n \Pi_{ji} d_j$. If this amount is less than zero, bank i has to pay nothing due to limited liability. If this amount is larger than the liabilities (\bar{d}_i), bank i must pay off its debt completely because of absolute priority. If the amount available is in the range from zero to \bar{d}_i , it is distributed proportionally amongst the debt holders. Given these restrictions we can therefore define a clearing payment vector.

Definition 1 A vector $d^* \in [\vec{0}, \bar{d}]$ is a **clearing payment vector** if

$$d^* = \left\{ \left[a^{NIB} + \Pi' d^* \right] \vee \vec{0} \right\} \wedge \bar{d}. \quad (3)$$

Alternatively, a clearing vector d^* can be characterized as a fixed point of the map $\Phi^1(\cdot; \Pi, \bar{d}, a^{NIB}, \Theta) : [\vec{0}, \bar{d}] \rightarrow [\vec{0}, \bar{d}]$ defined by

$$\Phi^1(d; \Pi, \bar{d}, a^{NIB}) = \left\{ \left[a^{NIB} + \Pi' d \right] \vee \vec{0} \right\} \wedge \bar{d}. \quad (4)$$

Φ^1 returns the minimum of the maximum possible payment and the promised payment \bar{d} . Hence, any supersolution $d \geq \Phi^1(d)$ is compatible with absolute priority but not necessarily with limited liability.

Eisenberg and Noe (2001) prove that a clearing vector exists for each realization of a^{NIB} . Under mild regularity conditions on the network structure the clearing vector is unique.

3 Estimating network exposures

How vulnerable a financial network is, depends on the capitalization of the banks and on the particular network structure which is given by the nonzero entries of L . Hence, an exact knowledge of all bilateral exposures would be extremely valuable. Unfortunately, this ideal data quality is rare.³ In almost all countries let alone across countries L can not be fully observed but has to be partially estimated from balance sheet data.

³The notable exceptions are Hungary (Lublóy (2005)) and Italy (Mistrulli (2007)).

Banks report regularly their total interbank claims and liabilities. The row and column sums of L are therefore known, at least in principle. Denote the vectors of row and column sums by b^r and b^c , respectively. It has to hold that

$$\sum_{i=1}^n b_i^r = \sum_{i=1}^n b_i^c = L^\Sigma$$

where L^Σ is size of the interbank market. Our own experience has shown that due to reporting errors and settlement modalities the sum of all interbank liabilities does not equal the sum of all reported interbank claims. This inconsistency has to be eliminated. Any adjustment of claims or liabilities to enforce the equality of aggregate claims and liabilities has to be accounted for in the non interbank part of the balance sheet of the affected banks.

We know that the diagonal elements of L have to be zero. Hence, for $n > 2$ we get $3n - 1$ constraints for the n^2 entries of L implying $n^2 - 3n + 1$ degrees of freedom for the estimation problem. The standard way in the literature to handle the estimation is to determine an admissible matrix L that minimizes the Kullback–Leibler divergence with respect to some specified nonnegative prior matrix U . We say that a matrix L is admissible if all entries are nonnegative, the column sums equal b^c , the row sums equal b^r , and the diagonal elements are zero. We denote the set of all admissible matrices by \mathcal{L} . The Kullback–Leibler divergence for nonnegative but otherwise arbitrary L is given by

$$D_{KL}(L, U) = \sum_{i=1}^n \sum_{j=1}^n \frac{l_{ij}}{L^\Sigma} \log \left(\frac{\frac{l_{ij}}{L^\Sigma}}{\frac{u_{ij}}{U^\Sigma}} \right) \quad (5)$$

where U^Σ is the sum of all entries in U .⁴ It is easy to verify that any minimizer of $D_{KL}(L, U)$ is also a minimizer of

$$D_{KL}^*(L, U) = \sum_{i=1}^n \sum_{j=1}^n l_{ij} \log \left(\frac{l_{ij}}{u_{ij}} \right). \quad (6)$$

and vice versa.

Suppose that the entries of L were drawn from a multinomial distribution with cell proba-

⁴The values of summands $0 \log(0)$ and $0 \log(0/0)$ are taken to be 0.

bilities $p_{ij} = u_{ij}/U^\Sigma$. The likelihood ratio statistic is then given by

$$\Lambda = \sum_{i=1}^n \sum_{j=1}^n l_{ij} \log \left(\frac{\frac{l_{ij}}{L^\Sigma}}{\frac{u_{ij}}{U^\Sigma}} \right).$$

Minimizing the Kullback–Leibler divergence is then equivalent to minimizing the likelihood ratio statistic with respect to the prior probabilities p_{ij} .

The estimate of L conditional on the prior matrix U is therefore given by

$$\hat{L}(U) = \arg \min_{L \in \mathcal{L}} D_{KL}^*(L, U). \quad (7)$$

In the application under consideration the set \mathcal{L} can be described by a set of linear constraints. We have to solve

$$\begin{aligned} \min_L \quad & D_{KL}^*(L, U) \\ \text{s.t.} \quad & \sum_{i=1}^n \sum_{j=1}^n a_{kij} l_{ij} = b_k \quad \text{for all } k \in \{1, \dots, K\} \\ & l_{ij} \geq 0 \quad \text{for all } i, j \in \{1, \dots, n\} \end{aligned} \quad (8)$$

where the K constraints are given by the row and column sums and the fact that the entries along the diagonal have to be zero. In principle any piece of information that can be described as a linear constraint on the elements of L can be taken into account. In particular, we may imbed prior knowledge about particular entries directly into the constraints. An extension incorporating inequality constraints which might be derived for instance from credit register data is straightforward. If credit register data are available this will usually pin down a large part of the entries in L .

The Langrangian of the minimization problem is given by

$$H(L, \lambda, \mu) = \sum_{i=1}^n \sum_{j=1}^n l_{ij} \log \left(\frac{l_{ij}}{u_{ij}} \right) - \sum_{k=1}^K \lambda_k (b_k - \sum_{i=1}^n \sum_{j=1}^n a_{kij} l_{ij}) - \sum_{i=1}^n \sum_{j=1}^n \mu_{ij} l_{ij}.$$

The Kuhn-Tucker conditions are therefore

$$\begin{aligned} \log \left(\frac{l_{ij}}{u_{ij}} \right) + 1 + \sum_{k=1}^K \lambda_k a_{kij} - \mu_{ij} &= 0 \quad \text{for all } i, j \in \{1, \dots, n\} \\ \lambda_k (b_k - \sum_{i=1}^n \sum_{j=1}^n a_{kij} l_{ij}) &= 0 \quad \text{for all } k \in \{1, \dots, K\} \\ \mu_{ij} l_{ij} &= 0 \quad \text{for all } i, j \in \{1, \dots, n\} \end{aligned}$$

Observe that the constraints are not necessarily compatible with an arbitrary prior matrix U . In applied work U is typically chosen as $u_{ij}^* = b_i^r b_j^c / L^\Sigma$ for $i \neq j$ and $u_{ij}^* = 0$ for $i = j$. Given that the reported data is accurate, this prior matrix is compatible with the constraints on the row and column sums and the zeros along the diagonal. A computationally efficient method to solve the minimization problem for U^* is Bregman's Balancing Procedure, described in Fang, Rajasekera, and Tsao (1997) which is also referred to as entropy projection method in the literature (Blien and Graef 1997).

Start the iteration by setting $l_{ij}^0 = u_{ij}^*$. Take the first constraint and set

$$l_{ij}^1 = \frac{l_{ij}^0 b_1}{\sum_{i=1}^n \sum_{j=1}^n a_{1ij} l_{ij}^0} \quad \text{for all } i, j \in \{1, \dots, n\}.$$

The iteration runs in a loop across the K constraints such that

$$l_{ij}^K = \frac{l_{ij}^{K-1} b_K}{\sum_{i=1}^n \sum_{j=1}^n a_{Kij} l_{ij}^{K-1}} \quad \text{for all } i, j \in \{1, \dots, n\}$$

and starts again with the first constraint

$$l_{ij}^{K+1} = \frac{l_{ij}^K b_1}{\sum_{i=1}^n \sum_{j=1}^n a_{1ij} l_{ij}^K} \quad \text{for all } i, j \in \{1, \dots, n\}.$$

If we set $\tau = (t \bmod K) + 1$, we may write

$$l_{ij}^{t+1} = \frac{l_{ij}^t b_\tau}{\sum_{i=1}^n \sum_{j=1}^n a_{\tau ij} l_{ij}^t} \quad \text{for all } i, j \in \{1, \dots, n\}.$$

The iteration is stopped as soon as some distance measure, e.g. the Euclidean distance, between L^t and L^{t+K} is smaller than a prespecified $\epsilon > 0$. Convergence of the procedure is shown in Fang, Rajasekera, and Tsao (1997).

The prior matrix U^* induces that the interbank claims are distributed as evenly as possible with respect to the constraints across all banks. The number of interbank linkages is thereby maximized. We get a complete network in the terminology of Allen and Gale (2000). Yet, such a complete network seems to be at odds with the available empirical literature on interbank linkages. Among others Upper and Worms (2004) and Cocco, Gomes, and Martins (2009) provide evidence that linkages are sparse and that the banking sector exhibits a tier structure. The

estimated matrix might thus be quite different from the actual matrix of interbank exposures.

Mistrulli (2007) studies the propagation of default within the Italian banking sector using the actual bilateral interbank exposures. He compares the results to those obtained when the liabilities matrix L is estimated by the above specified procedure. In Mistrulli's setup default leads to an exogenously given loss rate. The results indicate that using the estimated matrix underestimates contagious defaults for low loss rates and overestimates them for high loss rates. This result holds irrespective whether contagious defaults are weighted with total assets or not. Similar results were obtained by Degryse and Nguyen (2007).

On the theoretical side we know from the seminal contribution by Allen and Gale (2000) that the network structure matters. For symmetric banks a complete network absorbs shocks better than a sparse network. The insurance effect of a widely diversified portfolio dominates the contagion effect. Hence, the prior matrix U^* will underestimate the consequences of shocks which is in line with the findings for low loss rates by Mistrulli (2007).

Gai and Kapadia (2010) model the network as a directed random graph. They characterize the vulnerability of the network as a function of the capital buffer and the average degree, i.e. the average number of counterparties as a lender or as a borrower. Building on the work by Watts (2002) they show that (asymptotically) the probability of contagion is not monotone in the average degree. For capital buffers exceeding 4% the probability that at least 5% of the banks default due to contagion is highest for average degrees in the range of 3 to 6. For high average degrees contagion becomes a very rare phenomenon but if it happens, all banks default.

Gai and Kapadia (2010) assume that the interbank claims are evenly distributed across the counterparties. Amini, Cont, and Minca (2010) relax this assumption and refine the results of Gai and Kapadia (2010). Let $\mu(i, j)$ be the fraction of banks that have i obligors and j obligees within the network and let $p(i, j, 1)$ be the probability that a bank with i obligors and j obligees is vulnerable, i.e. dragged into default by the default of a single counterparty. Amini, Cont, and Minca (2010) define network resilience by

$$1 - \sum_{i,j} \frac{ij}{\lambda} \mu(i, j) p(i, j, 1) \quad (9)$$

where $\lambda = \sum_{i,j} j\mu(i, j)$. If network resilience is larger than zero and the fraction of initial defaults is sufficiently small then the probability is high that after the network is cleared only a small fraction of the banks is in default. On the contrary, if network resilience is smaller than

zero then there exists with high probability a subset of banks such that the default of any bank in this subset triggers the default of all other banks in the subset.

4 Creating loss scenarios

To measure systemic risk we need to expose the model of the banking system to some shocks and observe how these shocks propagate through the system causing banks to fail because of contagion. Heterogeneity in perspective on systemic risk as well as data availability explains the heterogeneity in modeling of shocks that we can observe in the literature. We will first in Section 4.1 survey some papers that look at a single bank's default, while keeping the financial position of all other banks unchanged. This way of modeling shocks is very useful for modeling purely idiosyncratic events such as a bank's default due to a rouge trader or operational loss. When we think of adverse macroeconomic developments as a cause of a systemic crisis we have to assume that all banks will be affected by the shock to a certain degree. The macroeconomic shock will then cause some banks to default while others are weakened and more prone to contagion. We will survey papers that use detailed information on the banks' exposures, mostly from regulatory filings, in Section 4.2 and examine how market information can be used to model loss scenarios in Section 4.3.

In this section we will present alternative ways of simulating shocks to the asset values a_{NIB} . We are not interested in the dynamics of the market value of liabilities since any firm is in default whenever it cannot honor its promised payments, i.e. pay the face value of the liabilities.

4.1 Idiosyncratic Bank Failures

Modeling idiosyncratic defaults can help to better understand the disruptions that one institution's failure would cause for the financial system. The perspective on financial stability underlying this modeling approach is whether the default of one bank might trigger the collapse of a considerable fraction of the banking system. Modeling idiosyncratic defaults is easy because no assumptions are required regarding the correlation structure of shocks to the banking system and are of practical importance when considering defaults due to large idiosyncratic bets by single banks, operational losses, losses from operations in a foreign market which only affect a

small number of banks that carry exposures to that foreign market, or losses from the default of a hedge fund with exposure to different risk factors than the average bank. With the exception of Elsinger, Lehar, and Summer (2006a), Elsinger, Lehar, and Summer (2006b) and Frisell, Holmfeld, Larsson, Omberg, and Persson (2007) most network models of contagion work with idiosyncratic failure scenarios.

4.2 Loss scenarios based on bank exposure data

Exposing the whole banking system to a shock has two main advantages relative to the modeling of single bank failures: first, the systemic risk analysis allows us to estimate not only the severity of contagion conditional on a bank's default but also the likelihood of each bank's default due to the outside shock as well as due to contagion in one consistent model. Second, macroeconomic shocks may have the potential to affect banks simultaneously and in the same way. As banks may have similar loan portfolios their balance sheets may deteriorate all at the same time. Banks face therefore more severe losses in the interbank market and through other contagion channels at times when they themselves suffer from above average losses. Ignoring the correlation in banks' asset portfolios thus severely underestimates the importance of contagion for systemic risk. The downside of modeling system wide shocks is that we require more assumptions on the structure of banks' asset correlations.

One stream of literature models bank loss scenarios based on regulatory filings or proprietary regulatory data. In some countries such as Germany, Italy, Spain, and Austria Central banks maintain loan registries with detailed information on banks' loan portfolios, often on a loan by loan basis. These data can be augmented with banks' reported exposures to interest rates, stocks, and foreign exchange prices. Such a rich dataset, which is mostly only available to bank regulators, also constitutes the main advantages of this approach to modeling bank losses. Therein also lies the greatest weakness of this modeling approach. National regulators often have limited information on their banks' foreign operations as well as foreign branches' risk exposures. Also reporting of derivatives is often based on notional values rather than risk factor exposures. Using regulatory filings is also particularly useful in countries where a significant fraction of the banking sectors' equity is not publicly traded. In several European countries cooperative banks, state owned or sponsored banks, and subsidiaries of international banks account for a significant fraction of the banking sector.

To simulate the systematic component of the shock most papers simulate first a macroeconomic scenario that affects all banks. Common variation in credit risk, which is often the most significant risk for banks, can be captured by scenarios of industry PDs. Gauthier, Lehar, and Souissi (2011) use draws from a distribution of industry PDs under a macro-stress scenario specified by the Bank of Canada and the IMF during one of its financial stability assessments. Elsinger, Lehar, and Summer (2006a) assume that average default rates are Gamma distributed with parameters estimated from historical data. Similarly other risk factors like foreign exchange rates and interest rates can be simulated conditional on the macroeconomic scenario.

Individual losses for banks can then be derived conditional on the macro-scenario. For credit risk, one draw of the macro scenario specifies the average PD per industry. Each of the bank's loans can then be seen as a Bernoulli distributed random variable that either gets repaid in full or defaults and only a fraction of the loan can be recovered. Total loan losses for each bank are then just the sum of outcomes of the Bernoulli variables and can be obtained via Monte Carlo simulation. For computational convenience Elsinger, Lehar, and Summer (2006a) and Gauthier, Lehar, and Souissi (2011) use a simplified CreditRisk+ model to derive each bank's distribution of loan losses conditional of the macro scenario and then take draws from this distribution to create loss scenarios.

4.3 Loss scenarios estimated from market data

An alternative way of modeling loss scenarios for banks is to estimate the dynamics of bank asset values using stock price data. This approach has the advantage that it can incorporate information beyond that contained in central bank reports by including the market's belief on the state of the banks. Stock prices reflect a bank's exposure to all risks including those not captured in central banks' reports. Market data is also available at a higher frequency allowing almost instantaneous assessments of financial stability. The problem with market data is that it not only includes the market's view on bank losses but also the market's view on regulatory action. Some banks may be perceived as less risky because they are too big to fail and therefore would be bailed out by the government in case of distress. Anticipated government intervention might thus contaminate exactly those observations that are most valuable for measuring systemic risk, the ones in the left tail of the distribution.

To generate bank loss scenarios most authors follow Merton (1973) who assumes that the

market value of the assets follows a geometric Brownian motion and interprets equity as a call option on assets with a strike price equal to liabilities. While there is no obvious choice for the maturity of debt, it can be interpreted as the time until the next audit of the bank, when the regulator can observe the asset value and close the bank if it is undercapitalized.

The maximum likelihood estimator developed by Duan (1994, 2000) allows to estimate the market values of banks' assets, their volatilities, the drift parameters and the correlation matrix from stock price data. Lehar (2005) uses this information to simulate scenarios for future asset values. To capture banks' systematic risk it is important to include the correlation structure in bank asset returns. Elsinger, Lehar, and Summer (2006b) combine these simulated scenarios with a network model to estimate systemic risk for the British banking system.

5 Clearing in the interbank market

Eisenberg and Noe (2001) develop an elegant algorithm to calculate clearing vectors, the fictitious default algorithm. It has the nice feature that it reveals a sequence of defaults. In the first round of the algorithm it is assumed that the payments made equal the promised payments \bar{p} . Banks that are not able to meet their obligations given their exogenous income and the payments they receive from other banks are determined. These banks default even if all of their interbank claims are honored. In the next step the payments of these defaulting banks are adjusted such that they are in line with limited liability. If there are no additional defaults the iteration stops. If there are further defaults the procedure is continued. The important point is that the algorithm allows to distinguish between defaults that are directly related to adverse economic situations – exogenous income – and defaults that are caused by the defaults of other banks.

More formally, define the diagonal matrix $\Lambda(d)$ by $\Lambda_{ii}(d) = 1$ if $a_i^{NIB} + \sum_{j=1}^n \Pi_{ji}d_j < \bar{d}_i$ and $\Lambda_{ii}(d) = 0$ otherwise. $\Lambda_{ii}(d) = 1$ if bank i 's exogenous income and the payments received from other banks do not suffice to repay the obligation \bar{d}_i . Define the map $d \rightarrow FF_{\hat{d}}(d)$ as follows:

$$FF_{\hat{d}}(d) \equiv \Lambda(\hat{d})[a^{NIB} + \Pi'(\Lambda(\hat{d})d + (\mathbf{I} - \Lambda(\hat{d}))\bar{d})] + (\mathbf{I} - \Lambda(\hat{d}))\bar{d}$$

This map returns for all nodes not defaulting under \hat{d} the required payment \bar{d} . For all other nodes it returns the node's value assuming that non defaulting nodes pay \bar{d} and defaulting nodes pay d . Under suitable restrictions this map has a unique fixed point which is denoted by $f(\hat{d})$.

Note that the equation for the fixed point

$$f(\hat{d}) = \Lambda(\hat{d})[\Pi'(\Lambda(\hat{d})f(\hat{d}) + (\mathbf{I} - \Lambda(\hat{d}))\bar{d}) + a^{NIB}] + (\mathbf{I} - \Lambda(\hat{d}))\bar{d}$$

can actually be written quite compactly as

$$[\mathbf{I} - \Lambda(\hat{d})\Pi'\Lambda(\hat{d})](f(\hat{d}) - \bar{d}) = \Lambda(\hat{d})(a^{NIB} + \Pi'\bar{d} - \bar{d}). \quad (10)$$

Premultiplying by $(\mathbf{I} - \Lambda(\hat{d}))$ yields

$$(\mathbf{I} - \Lambda(\hat{d}))(f(\hat{d}) - \bar{d}) = \vec{0}.$$

For banks that do not default $\Lambda_{ii}(\hat{d}) = 0$ and $f_i(\hat{d}) = \bar{d}_i$. Premultiplying (10) by $\Lambda(\hat{d})$ gives

$$\Lambda(\hat{d})(\mathbf{I} - \Pi')\Lambda(\hat{d})(f(\hat{d}) - \bar{d}) = \Lambda(\hat{d})(a^{NIB} + \Pi'\bar{d} - \bar{d}).$$

The ij th entry of $\Lambda(\hat{d})(\mathbf{I} - \Pi')\Lambda(\hat{d})$ is zero unless $\Lambda_{ii}(\hat{d}) = \Lambda_{jj}(\hat{d}) = 1$. To calculate the fixed point, it suffices to consider the subsystem (submatrix) of defaulting nodes. The original system of equations can be chopped up into two independent systems. This is a major advantage if the number of nodes is large and default is a rare event.

Eisenberg and Noe (2001) show that under the assumption that $a^{NIB} > \vec{0}$ the sequence of payment vectors $d^0 = \bar{d}$, $d^i = f(d^{i-1})$ decreases to a clearing vector in at most n iterations. The assumption that $a^{NIB} > \vec{0}$ is essential as is illustrated by the following example.

Example 1

$$a^{NIB} = \begin{pmatrix} 1 \\ \frac{3}{4} \\ -\frac{9}{8} \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}, \quad \bar{d} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Setting $d^0 = \bar{d}$ yields

$$a^{NIB} + \Pi'd^0 = \begin{pmatrix} \frac{9}{4} \\ \frac{3}{2} \\ -\frac{1}{8} \end{pmatrix}, \quad \Lambda(d^0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad d^1 = f(d^0) = \begin{pmatrix} 1 \\ -\frac{3}{20} \\ -\frac{6}{5} \end{pmatrix}.$$

Hence, d^1 is not feasible and the algorithm breaks down. A possible remedy would be to

use $d^i = [f(d^{i-1}) \vee \vec{0}]$. This procedure results in $d^2 = d^1 = (1, 0, 0)'$. It is easy to verify that $(1, 0, 0)'$ is not a clearing vector. The unique clearing vector for the example is given by $d^* = (1, \frac{3}{4}, 0)'$.⁵

In cases where the fictitious default algorithm does not work anymore, e.g. in the case of a detailed seniority structure or if $a^{NIB} \not\geq \vec{0}$, it is still possible to define a simple but admittedly less elegant iterative procedure to calculate a clearing vector. We start with $d^0 = \bar{d}$ and calculate $a^{NIB} + \Pi' d^0$. If d^0 is affordable, i.e. $d^0 \leq [(a^{NIB} + \Pi' d^0) \vee \vec{0}]$, we are done. Otherwise the payments are reduced such that they are in line with limited liability, i.e. $d^1 = [(a^{NIB} + \Pi' d^0) \vee \vec{0}] \wedge \bar{d}$. The iterative procedure $d^{i+1} = [(a^{NIB} + \Pi' d^i) \vee \vec{0}] \wedge \bar{d}$ started at $d^0 = \bar{d}$ is well defined, decreasing, and converges to the largest clearing vector d^+ .

Both solution algorithms yield the same sequence of defaults. Banks defaulting in the first round are those that default even if their claims are honored fully. Banks defaulting in later rounds are dragged into default by their interbank counter parties.

6 Empirical findings

The ideas discussed in the previous sections have been applied by Elsinger, Lehar, and Summer (2006a) and Elsinger, Lehar, and Summer (2006b) to a dataset of Austrian banks as well as to banks in the UK. While the Austrian data set was very detailed the UK dataset was rather coarse. The detailed dataset allows the description of the actual bank balance sheets at a high level of resolution. Such data are not widely available. In particular loan registers do not exist in all countries. The paper Elsinger, Lehar, and Summer (2006b) was written mainly to show that the basic ideas of contagion analysis can still be applied when these detailed data are not available. In this section we report on the results obtained from the Austrian data. Not only are these data very detailed but the regular application of the model simulations at the Austrian Central Bank allows us with some confidence to make robust statements about the main empirical findings of this research.

In a nutshell the empirical findings show two things:

1. Contagion of insolvency due to interbank exposures is rare.

⁵The fictitious default algorithm works if each d^i is a supersolution. This can be guaranteed for $a^{NIB} > \vec{0}$ and $d^0 = \bar{d}$. For $a^{NIB} \not\geq \vec{0}$ this property may not hold.

2. It seems very hard to create quantitatively realistic scenarios that will lead to a significant amount of contagion.

The first finding has been discovered in numerous simulation studies on interbank markets around the world (see (Upper 2011) for an excellent survey). The second finding seems to be reported mainly in Elsinger, Lehar, and Summer (2006a) and Elsinger, Lehar, and Summer (2006b). This has to do with the fact that these papers build simulations that do not rely on idiosyncratic hypothetical failure scenarios for individual institutions but work with a simulated profit and loss distribution for the entire banking portfolio. In the following we discuss the main empirical findings in Elsinger, Lehar, and Summer (2006a). We then give an overview of the findings from other simulation studies and finally discuss some general issues.

6.1 Contagion is likely to be rare

The simulation of scenarios and the resulting portfolio losses in combination with the network clearing algorithm for interbank holdings allows for a decomposition of insolvencies into cases resulting directly from shocks and cases that are indirect consequences of the failure of some other institution to which the failing bank is linked by interbank debt. This decomposition contains information about the likely significance of domino effects. In the simulations two different clearing situations are considered: In the first situation labeled the "short run" it is assumed that following a default there will be no payments between banks after netting. The second situation labeled the "long run" assumes that the residual value after a default can be fully transferred to the creditor institution. The main finding of this decomposition is that among all simulated 100.000 scenarios contagious defaults will be observed only in 0.86% of all simulated scenarios in the short run and in 0.05% in the long run. In all other scenarios the defaults occur in the simulation as a direct consequence of the risk factor moves. While contagion is rare there are scenarios where a large number of contagious default occurs. Since the publication of Elsinger, Lehar, and Summer (2006a) a contagion model based on the ideas in this paper has been implemented at the Austrian Central Bank (Boss, Krenn, Pühr, and Summer (2006)) and is used by financial stability analysts. Data are updated every quarter and regular simulations are run. The basic fact found in the paper that the majority of defaults in the simulation comes from direct exposure to risk factors and that domino effects of contagion are likely to be rare turned out to be a robust feature of the simulations over time. This fact remained robust even when data from the financial crisis entered the simulation.

6.2 Domino effects are likely to occur only in doomsday scenarios

In Elsinger, Lehar, and Summer (2006a) the question under which circumstances contagion becomes widespread is discussed by introducing the assumption that in a default assets are partially destroyed due to bankruptcy costs. It is found that there is little contagion for low bankruptcy costs but as total assets are destroyed up to an amount of 30% and more the number of contagious defaults increases sharply.

A look at the loss distributions resulting from simulations of losses due to market and credit risk shows that in 2006 even the extreme quantiles of the loss distribution for market risk would destroy only about 1.6% of total assets in the entire system in the case of market risk and about 1% in the case of credit risk. Even if the losses were much worse we were not near the threshold of 30% above which domino effect contagion becomes a significant issue.

6.3 What are the findings of other simulation studies?

Most simulation studies of interbank contagion in contrast to Elsinger, Lehar, and Summer (2006a) have worked with hypothetical single bank failures and an exogenous recovery rate from interbank lending. These studies are systematically surveyed in Upper (2011). They include Amundsen and Arndt (2011), Martin Blavarg (2002), Degryse and Nguyen (2007), Furfine (2003), Gueroo-Gómez and Lopez-Gallo (2004), Lublóy (2005), Mistrulli (2007), Müller (2011), Sheldon and Maurer (1998), Toivanen (2009), Upper and Worms (2004), Lelyveld and Liedorp (2006), Wells (2004).

While most of these studies come to the conclusion that contagion is likely to be rare they do not reach a clear cut result about the potential impact of contagion. While some studies found that in the worst case domino effects could destroy between 15% up to 20% of the banking system others found very little possibility for contagion. An exception is a study based on Swedish data ((Frisell, Holmfeld, Larsson, Omberg, and Persson 2007)) using aggregate shocks to bank portfolios along the lines of Elsinger, Lehar, and Summer (2006b). They find a high probability of domino effects, occurring in approximately one half of the cases in which one of the top 4 Swedish banks fail. In summary these studies find that domino effects are unlikely but when they occur they may affect a substantial part of the banking system.

6.4 An alternative interpretation of the empirical findings

The papers discussed here were all written before the financial crisis. None of these models predicted the crisis and they played no major role in policy decisions that were taken during the crisis. So were there models a failure in practice? Here we would like to give a more positive interpretation.

The empirical findings of this research help to refocus the research agenda on systemic risk. This is because they settle the issue how important domino effects of insolvency working through the balance sheet mechanics of the banking system really are. The answer is that they are just not very important.

In the time before the financial crisis many institutions were preoccupied with domino stories of systemic risk. But what we learn from the research is that there is no way that the losses from the US subprime crises that were predicted by the IMF in April 2008 of 945 billion dollars (International Monetary Fund (2008)) – while huge as an absolute number – would be able to bring down the world financial system, certainly not through domino effects resulting from such a loss. The dynamics of a financial crisis come from powerful amplification mechanisms that have to do with the interaction of behavior and the pricing of risk (see Shin (2010)). The crisis showed and reminded us quite clearly that the core mechanisms at work in the build up and the unfolding of a financial crisis lie in the interaction between leverage, asset prices and portfolio decisions. Models of domino effects arising from the balance sheet mechanics of a complicated web of interbank debt can be extended to take these dynamics into account.

The qualitative aspects of the amplification mechanisms were analyzed and described in quite a few books and papers that have been published since the financial crisis. These papers include among others Shin (2010), Geanakoplos (2009), Kiyotaki and Moore (2008), Brunnermeiner and Pedersen (2009), Brunnermeier (2009), Holmström and Tirole (2011), Hellwig (2008). How the insights of this literature can enter the quantitative analysis of systemic risk is still an open and largely unresolved issue.

7 Extensions

Several recent papers use a systemic risk model as presented in this paper to measure not only the risk of the whole system but also to identify the contributions of individual banks to overall risk. Existing studies use stock returns (Adrian and Brunnermeier (2011), Acharya, Pedersen, Philippon, and Richardson (2011), or Billio, Getmansky, Lo, and Pelizzon (2011)) or a network model (Drehmann and Tarashev (2011)).

In the aftermath of the financial crisis there has been a stronger interest to adjust bank capital requirements to better reflect not only the risk of the bank itself but also the bank's contribution to the risk of the system. However, using risk contributions for regulation is not straightforward. Once bank capital requirements are set to the risk contributions, each bank's PD and default correlation will change, the risk of the whole banking system will adjust, and therefore each bank's risk contribution will not be the same any more. Gauthier, Lehar, and Souissi (2011) follow a fixed point procedure to compute macroprudential capital requirements such that each bank's capital requirement equals its risk contribution.

David and Lehar (2012) extend the basic network model as it is presented here by allowing banks that cannot honor their interbank obligations in full to renegotiate their payments. They model a multi player bargaining game in which banks that are unable to pay their obligations d to the interbank market can renegotiate for a lower payment with their counter-parties. They show that systemic risk drops once renegotiation is allowed and that banks optimally choose an ex-ante fragile network to eliminate renegotiation failure.

Other extensions might be based on recent suggestions by Brunnermeier and Krishnamurthy (2011). They stress the importance to overcome the pure balance sheet mechanics approach to systemic risk measurement. As we have discussed in 6.4 it is crucial to include the behavioral responses of market participants to shocks. Brunnermeier and Krishnamurthy (2011) make a proposal of how a database could be built to elicit risk and liquidity sensitivities of market participants. These data could then be used to extend the network model by including behavior. How this can be done conceptually is for instance demonstrated in the Bank of England's RAMSI model (see Alessandri, Gai, Kapadia, Mora, and Pühr (2009)).

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